Radiation from a D-dimensional collision of shock waves: numerical methods

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nr/hep^2 - IST

Acknowledgments

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  Flávio Coelho, Carlos Herdeiro & Carmen Rebelo

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- **nr/hep^2 organizers**

Motivation I – The Transplanckian Problem

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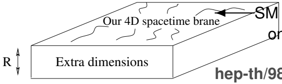
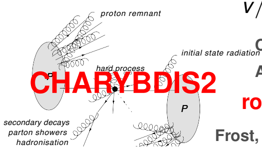
Transplanckian scattering @ $\sim 1\,\mathrm{TeV} \Rightarrow$ BHs @ LHC



SM effective theory
on a thin brane

hep-th/9803315 (ADD)



CHARYBDIS2

$v/c > 0.999$ **@ LHC**

CMS arXiv:1012.3357

ATLAS-CONF-2011-065

rough model Gr. Rad.

Frost, MS, Gaunt et. al. 0904.0979

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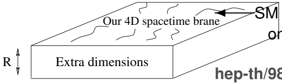

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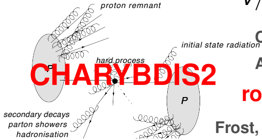

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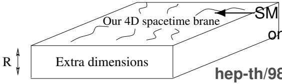

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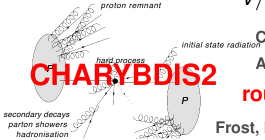

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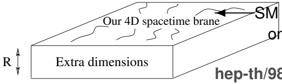

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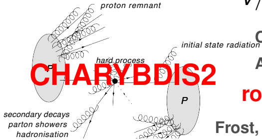

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   - In $D \geq 4$, $\epsilon_{\rm rad}^{(1)} = ?$, $\epsilon_{\rm rad}^{(2)} = ?$

Motivation II – Numerics vs Analytics

This problem is also technically very rich

1. Analytically:
   - Allows a consistent setup of an evolution problem with well defined *initial conditions*,
   - An *approximate integral solution* can be obtained using perturbation theory,
   - Requires the discussion of a general radiation extraction formula in axially symmetric spacetimes at null infinity.
2. Numerically it requires:
   - The integration of many non-trivial integral solutions using Green functions,
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$\Rightarrow$ Excelent ground to illustrate **many** techniques, within a well defined and **interesting problem.**

Plan for Today

1. AS Gravitational Shock Waves
   - Physical properties & Ray Optics
   - Superposition & causal structure
   - Perturbative Setup & Solutions

2. Numerical Strategies – Surface Integrals
   - Breaking down the problem
   - Numerical domain search
   - Dealing with the Integrand

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Tomorrow: Radiation extraction, Sources & Results.

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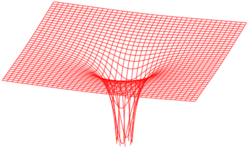
$$ds^2 = -\left(1 - \frac{\mu}{r^{D-3}}\right) dt^2 + \left(1 - \frac{\mu}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2$$



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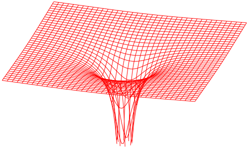
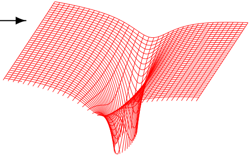


Boost along z



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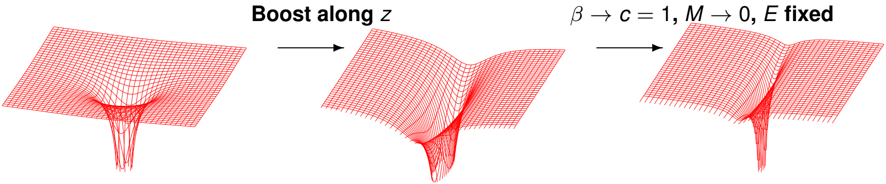

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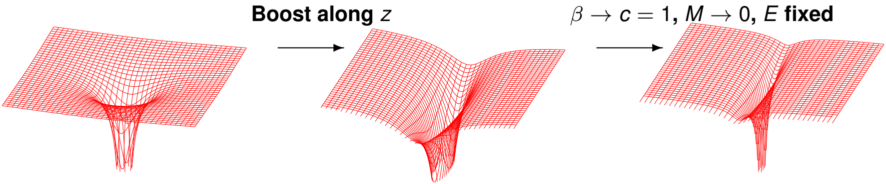

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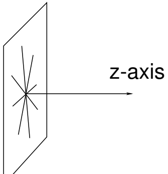

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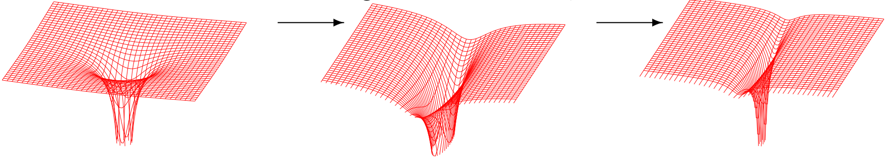
Boost along z $\qquad$ $\beta \rightarrow c = 1$, $M \rightarrow 0$, E **fixed**



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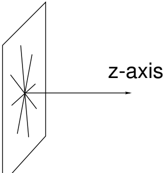
Flat region II $\qquad$ **Flat region I**



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Basic properties of a single shock wave I

- Solution of Einstein's equations, **point source** $P^\mu = E\, n^\mu$

$$T^{\mu\nu} = E\delta(u)\delta^{(D-2)}(x^i)n^\mu n^\nu \quad , \quad n^\mu n_\mu = 0 \quad , \quad \kappa = \frac{8\pi G_D}{\Omega_{D-3}}E$$

- The **shock profile** is $\Phi(\rho) = \begin{cases} \frac{2}{(D-4)\rho^{D-4}} \ , & D > 4 \\ \end{cases}$.

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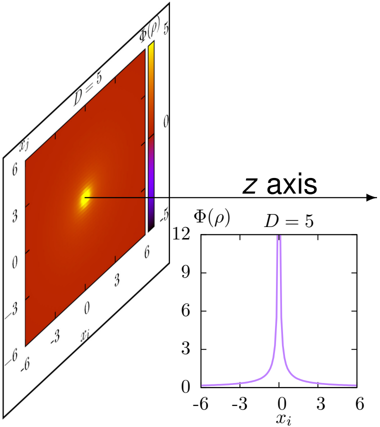

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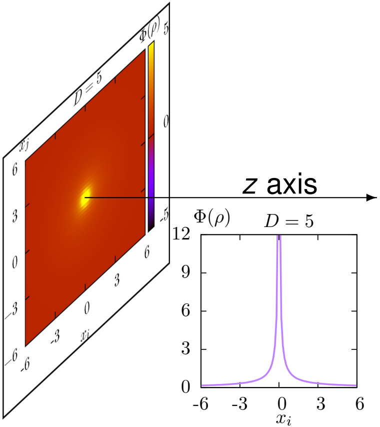

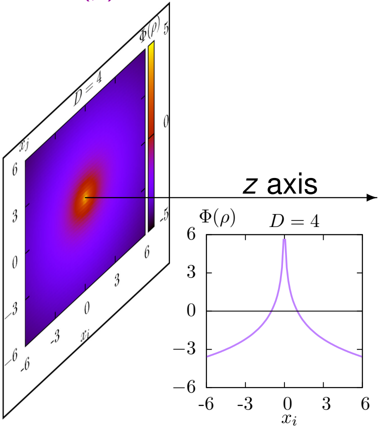

Basic properties of a single shock wave II

Symmetries:

- Axial symmetry (ϕ_i rotations on $d\Omega_{D-3}$)
- Advanced null translations $v \to v + const.$

$\Rightarrow$ Metric of **one shock** wave effectively **2D** (u, ρ)

- Boosts along $+z$, with velocity $\beta \equiv \tanh \alpha$, up to a scalling

$$E \to E' = e^{\alpha} E \qquad (\Leftrightarrow \kappa \to \nu = e^{\alpha} E)$$

Geometrical optics:

- **Riemann** tensor singular on the shock plane
- **Null** rays & tangent vectors discontinuous across shock

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Null rays scattering through shock wave $D = 5$

Play Video 1

Play Video 2

- Null rays more delayed if aimed closer to center
- Also more bent inwards, (even backwards) closer to center
- Envelope of outermost null rays defines causal boundary & becomes original null plane of rays @ late times
- Envelope of innermost rays defines a sphere @ late times

Null rays scattering through shock wave $D = 4$

Play Video 1 Play Video 2

Same as before except that:

- Rays far from center advance instead of delaying (coordinate effect since jump $\propto \Phi(\rho)$).
- But deflection/convergence of rays produces same effects (physical effect since $\propto \frac{d\Phi}{d\rho}$)

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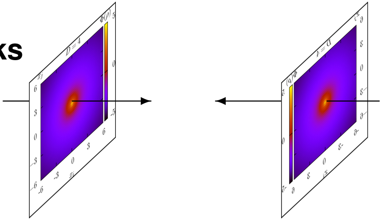
Superposition of two shock waves in CM

Consider oppositely moving shocks

$\Rightarrow$ <u>same line element with $u \leftrightarrow v$!</u>



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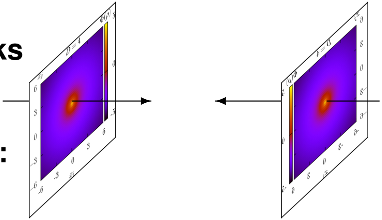

Superimposing 2 shocks we know:

1. They move @ speed of light

   $\Rightarrow$ *Cannot influence each other before collision*

   $\Rightarrow$ *Metric before collision is sum of two shock metrics*

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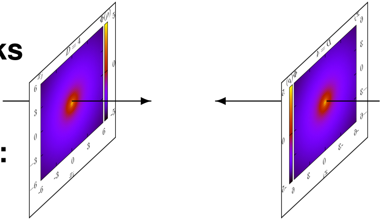

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2. How null generators of a shock scatter through other shock
   $\Rightarrow$ *Defines causal structure, i.e. regions causally disconnected to collision (see next slide).*

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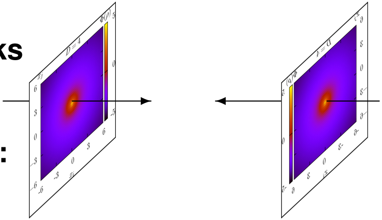

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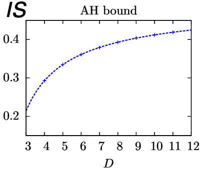
3. There is an apparent horizon already before the collision
   $\Rightarrow$ *Black hole must form & Bound on inelasticity is*

$$\epsilon_{\text{radiated}} \leq 1 - \frac{1}{2}\left(\frac{D-2}{2}\frac{\Omega_{D-2}}{\Omega_{D-3}}\right)^{\frac{1}{D-2}}$$



D. M. Eardley and S. B. Giddings, gr-qc/0201034

Causal structure $D = 5$

When summing metrics before collision:

- Easier to interpret metric in coordinates (u, v, ρ) adapted to one shock wave (e.g. right moving $u = 0$), since flat.

Play Video

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- $ds^2 = ds^2_{u,flat} + ds^2_{v,scatter}$, where $ds^2_{v,scatter}$ is flat, at points further away from center than envelope of 1st null rays.

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Causal structure $D = 5$

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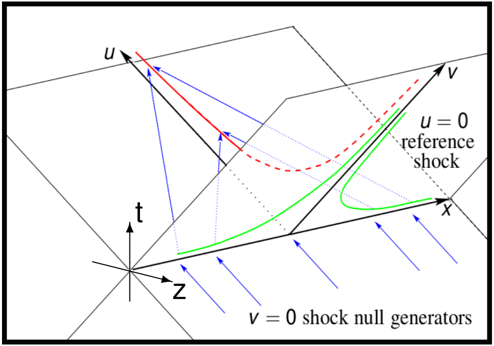

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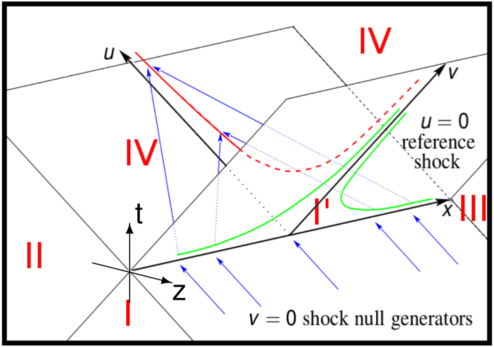

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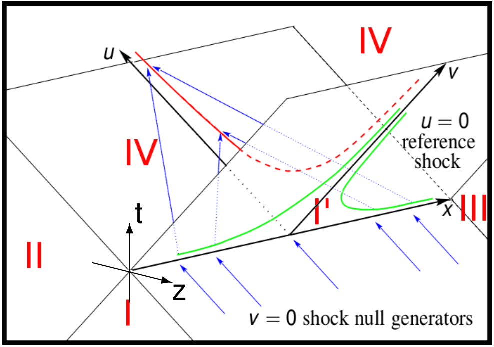

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Causal structure $D = 4$

Similar diagrams except that:

- Profile function $\lim_{\rho \to +\infty} \Phi(\rho) = -\infty$.
- Rays with $\rho > \kappa \Rightarrow$ step forward in time.

Play Video 1

Play Video 2

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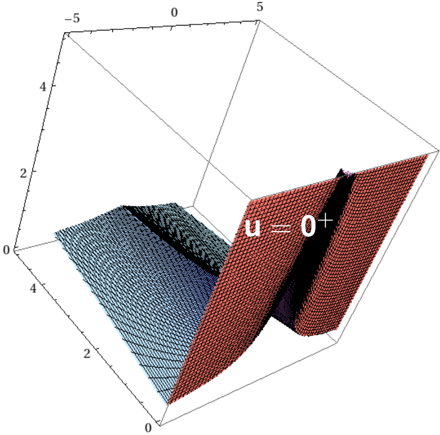
Play Video 1

Play Video 2

<u>Note</u> (**all** *D*): backscattered rays ($\frac{\rho_{incident}}{\kappa^{1/(D-3)}} < 1$) **inside AH** $\Rightarrow$ Trapped!

Exact initial conditions in Brinkmann coordinates

In these (asymmetric) coordinates adapted to shock $u = 0$:



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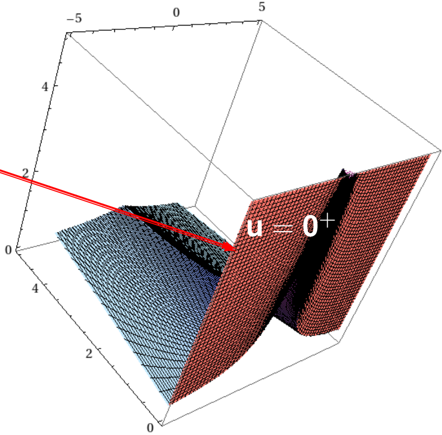

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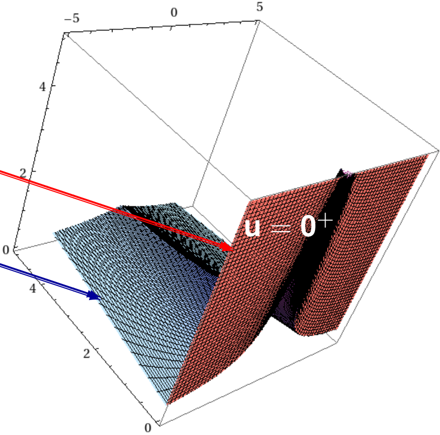

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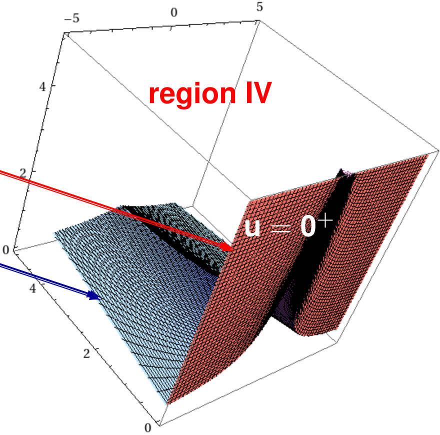

$\Rightarrow$ Well defined evolution problem into (future) **region IV**, given **EXACT** initial conditions.

Outline

1. AS Gravitational Shock Waves
   - Physical properties & Ray Optics
   - Superposition & causal structure
   - Perturbative Setup & Solutions
2. Numerical Strategies – Surface Integrals
   - Breaking down the problem
   - Numerical domain search
   - Dealing with the Integrand

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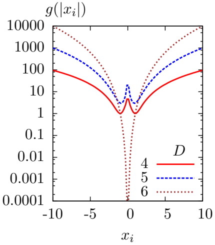

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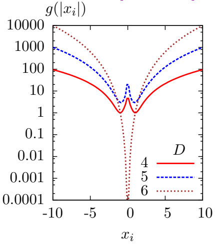

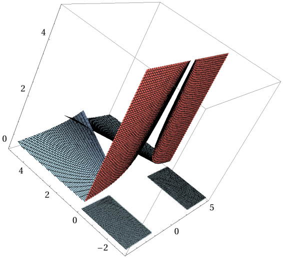

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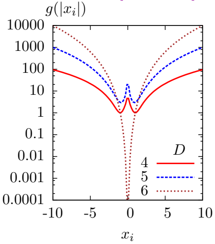

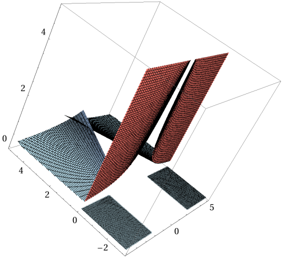

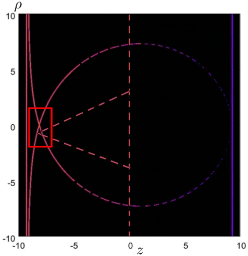

Two roads to a perturbative construction - Road I

D'Eath and Payne's Physical picture:

1. Perform a large boost along z with velocity $\beta \equiv \tanh \alpha$
2. Energy of $u = 0$ shock $\kappa \to \nu = e^{\alpha}\kappa$, blue-shifted.
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$\Rightarrow$ Weak shock ($v=0$) small **perturbation of flat space-time.**

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In Boosted frame

$\Rightarrow$ *Smaller* $\rho_{incident}$ *allowed*.

$\Rightarrow$ *Perturbative region* wider (expands away from the axis).

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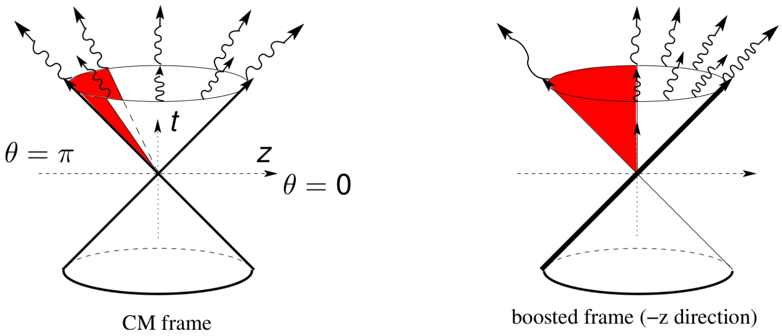

Boost emphasizes validity of approximation close to the axis.

Perturbative expansion

In the perturbative region (close to rays originating from perturbative initial conditions):

- Assume perturbative ansatz

$$g_{\mu\nu}(u>0,v,x_i) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \kappa^n h^{(n)}_{\mu\nu}$$

- Fix gauge order by order (de Donder condition $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$)

$$x^\mu \to x^{N\mu} = x^\mu + \sum_{n=1}^{\infty} \kappa^n \xi^{(n)\mu}$$

- Obtain **decoupled tower of wave equations with source**

$$\Box h^{(n)N}_{\mu\nu} = T^{(n-1)}_{\mu\nu}\left[h^{(k<n)}_{\alpha\beta}\right] \, .$$

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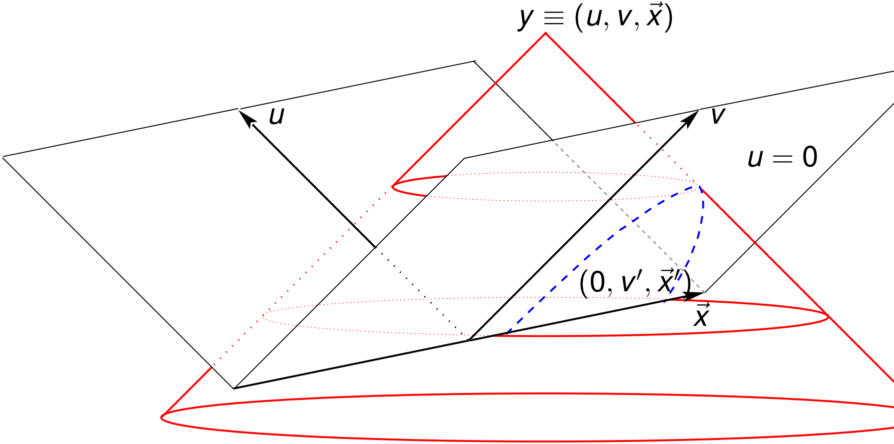
Causal structure of the background & formal solution



$$h^{(n)N}_{\mu\nu}(y) = \int_{u'>0} d^D y' \, G(y, y') \left[T^{(n-1)}_{\mu\nu}(y') + 2\delta(u')\partial_{v'} h^{(n)N}_{\mu\nu}(y') \right]$$

Summary of perturbative solutions I

Axial symmetry allows to expand all metric elements using

$$\Gamma_i \equiv \frac{x_i}{\rho} \ , \quad \delta_{ij} \ , \quad \Delta_{ij} \equiv \delta_{ij} - (D-2)\Gamma_i\Gamma_j \ ,$$

[1] $m = 0, 1, 2$ is the rank of the transverse space-tensor.

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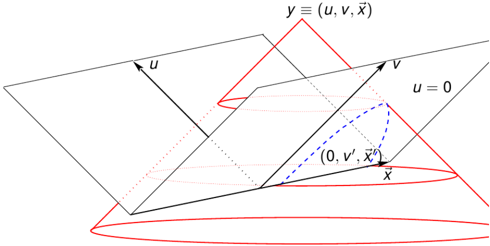

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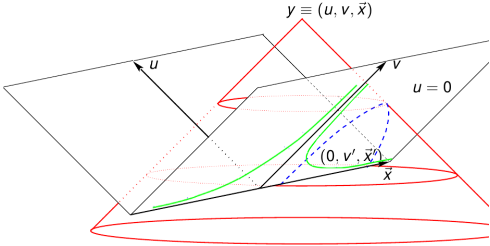

Physical interpretation of the solutions

A very clear physical picture to this tower of solutions is:

1. At 1st order, there is an impulsive creation of a configuration of radiation on $u = 0^+$ in the initial conditions. This propagates freely on the flat background.
2. The 2nd order $h^{(2)}_{\mu\nu}$ propagates on a flat background, and another signal of radiation is generated by the source of $h^{(1)}_{\mu\nu}$. This must encode backscattering.
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Main code written in C++ with GSL libraries for numerical integration & bracketing algorithms.

I will show Mathematica checks/examples to illustrate.

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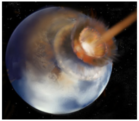

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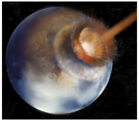

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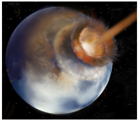

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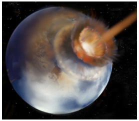

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Be patient, take your time to code. It saves you time!

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Domain conditions

Consider the two following polynomials/functions ($s = \pm$)

$$C_s(y) \equiv y^{D-2} + 2sy^{D-3} - (\sqrt{2}qp - 1)y^{D-4} + \sqrt{2}qy^{D-4}\psi(y)$$

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$\Rightarrow$ Domain determination reduces **to root finding**

(see Mathematica example 1)

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Sensitive features of the integrand $I_m^{D,n}(x_\star)$

Integrand contains several pieces **to be computed with care**:

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   Example of *task* to be done *once at the begining.*
   (see Mathematica example 2.1)

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Summary

So far we have seen:

- How to superimpose two shock waves & write exact initial conditions for evolution;
- Formulated approximate perturbative scheme to find gravitational perturbations close to axis;
- Found the general perturbative solutions, order by order, as source and volume integrals;
- Addressed surface integrals with some practical examples.

Tomorrow $\Rightarrow$ Numerical results, radiation extraction & Higher orders.